Math 1510J Week 1

Set

A set is a collection of elements

eg A = the set of the first 4 positive even numbers

set

$= \{2, 4, 6, 8\}$

elements

Notations

$x \in A$ means x is an element of A

$x \notin A$ means x is not an element of A

$A \subseteq B$ means A is a subset of B

(ie. every element of A is an element of B)

$A \nsubseteq B$ means A is not a subset of B.

eg $A = \{2, 4, 6, 8\}$ $B = \{2, 8\}$ $C = \{2, 4, 6\}$

$8 \in A, B$ but $8 \notin C$

$B \subseteq A$, $A \nsubseteq B$, $B \nsubseteq C$, $C \nsubseteq B$

Set Operations

Suppose A, B are sets. Then

Intersection

such that

$$A \cap B = \{x : \underbrace{x \in A \text{ and } x \in B}_{\text{condition}}\}$$

the set of all x such that $x \in A$ and $x \in B$

Union

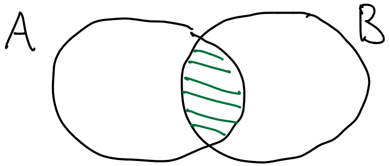
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Relative complement of B in A

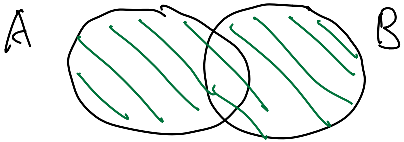
$$A \setminus B = \{x \in A : x \notin B\}$$

the set of all $x \in A$ such that $x \notin B$

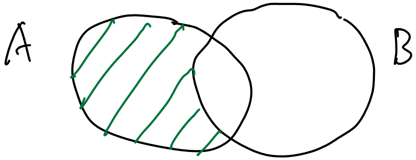
Picture (Venn Diagram)



$A \cap B$



$A \cup B$



$A \setminus B$

eg $A = \{2, 4, 6\}$ $B = \{3, 6, 9\}$

$A \cap B = \{6\}$

$A \cup B = \{2, 3, 4, 6, 9\}$

$A \setminus B = \{2, 4\}$

Some important sets

$\mathbb{N}$ = the set of all natural numbers

$= \{1, 2, 3, 4, \ldots\}$

$\mathbb{Z}$ = the set of all integers

$= \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q}$ = the set of all rational numbers

$= \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$

$\mathbb{R}$ = the set of all real numbers

eg $\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$

$\frac{3}{2} \in \mathbb{Q}$, $\frac{3}{2} \notin \mathbb{Z}$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Intervals Let $a, b \in \mathbb{R}$

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$ open interval

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ closed interval

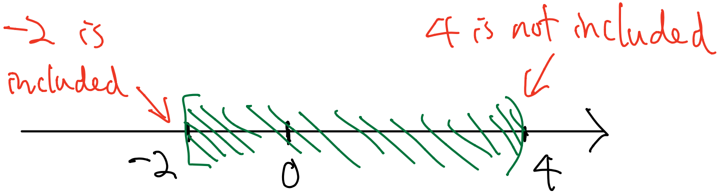
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$

$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$

$(-\infty, \infty) = \mathbb{R}$

eg $[-2, 4)$



Function

Let A, B be sets.

A function $f : A \to B$ is a rule of assigning to each element of A an element of B

A = Domain of f

B = codomain of f

$\operatorname{Im} f$ = image of f
= range of f
$= \{f(x) \in B : x \in A\}$
= set of outputs

Other notations

D_f = Domain of f

$\operatorname{Im} f = f(A)$ = range of f

eg

domain, codomain

function → $f : \mathbb{R} \to \mathbb{R}$

$$\underbrace{f(x) = x^2 + 3}_{\text{"rule of assignment"}}$$

$$f(3) = 12$$

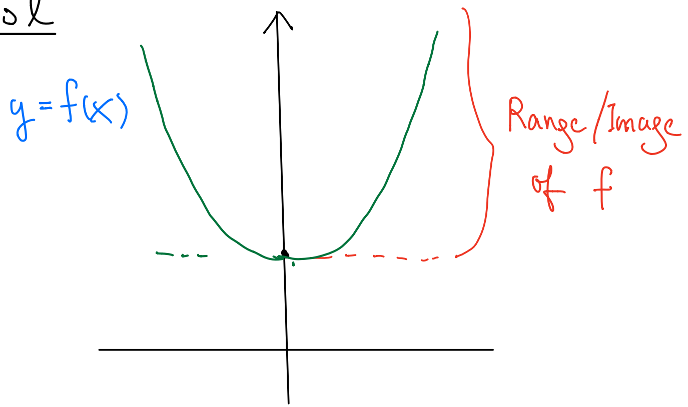
input output

Q What is the range of f?

Can consider the graph of f

or do the algebra

Sol



Note $x^2 \geq 0$ (x^2 can be any number ≥ 0)

$\Rightarrow \quad x^2 + 3 \geq 3$

$\Rightarrow \quad f(x) \geq 3$

$\operatorname{Im} f = [3, \infty)$

Implied domain

If a function $f(x)$ is given by an expression without specifying its domain, then the domain will be assumed to be the largest subset of $\mathbb{R}$ such that the expression makes sense.

That domain is called the implied domain (natural domain)

Useful rules

1. Denominator $\neq 0$

2. For $\log(g(x))$, need $g(x) > 0$

3. Let m be an positive even number

   For $\sqrt[m]{h(x)} = [h(x)]^{\frac{1}{m}}$

   Need $h(x) \geq 0$

Rmk If m is odd,

$[h(x)]^{\frac{1}{m}}$ makes sense even if $h(x) < 0$

$$(-64)^{\frac{1}{3}} = -4 \checkmark$$

$$(-64)^{\frac{1}{2}} \times$$ (not a real number)

eg Find implied domain of

a. $\log(x^2 - 3x - 10)$

b. $\dfrac{x-3}{\sqrt[4]{3-|x|}}$

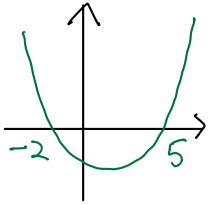
c. $(x+2)^{\frac{2}{3}}$

d. $f-g$, where $f(x) = \dfrac{1}{1+x}$ $g(x) = \dfrac{1}{1-x}$

Sol

a. Need $x^2-3x-10>0$

$(x-5)(x+2)>0$



$\Rightarrow$ $x>5$ or $x<-2$

$\Rightarrow$ Implicit domain $=(-\infty,-2)\cup(5,\infty)$

b. Need $3-|x|\geq 0$ under $\sqrt[4]{\quad}$

Also, $\sqrt[4]{3-|x|}\neq 0$

$\Rightarrow$ $3-|x|>0$

$\Rightarrow$ $3>|x|$

$\Rightarrow$ $-3<x<3$

$\Rightarrow$ Implicit domain $=(-3,3)$

c. $(x+2)^{\frac{2}{3}}=\sqrt[3]{(x+2)^2}$

3 is odd number $\sqrt[3]{h(x)}$ is defined for any value of $h(x)$

Implied domain $=(-\infty,\infty)=\mathbb{R}$

d. $(f-g)(x)=f(x)-g(x)$

$$=\frac{1}{\underbrace{1+x}_{x\neq -1}}-\frac{1}{\underbrace{1-x}_{x\neq 1}}$$

$\Rightarrow$ Implied domain $=\mathbb{R}\setminus\{\pm 1\}$

$=(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

Operation on functions

Let $f(x), g(x)$ be functions. Then

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\, g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$(g \circ f)(x) = g(f(x))$$ (Composition)

$$x \xmapsto{\ f\ } f(x) \xmapsto{\ g\ } g(f(x))$$

$$D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g$$

$$D_{f/g} = (D_f \cap D_g) \setminus \{x \in D_g : g(x) = 0\}$$

$$D_{g \circ f} = \{x \in D_f : f(x) \in D_g\}$$

eg Let $f(x) = x^2 - x$, $g : (2, \infty) \to \mathbb{R}$

a. Find $(f \circ f)(-1)$

b. Find the implied domain of $g \circ f$

Sol

a. $(f \circ f)(-1) = f(f(-1)) = f(2) = 2$

b. $(g \circ f)(x) = g(f(x))$

$$= g(x^2 - x)$$

$$D_g = (2, \infty) \Rightarrow x^2 - x \in (2, \infty)$$

$$\Rightarrow x^2 - x > 2$$

$$\Rightarrow x^2 - x - 2 > 0$$

$$\Rightarrow (x-2)(x+1) > 0$$

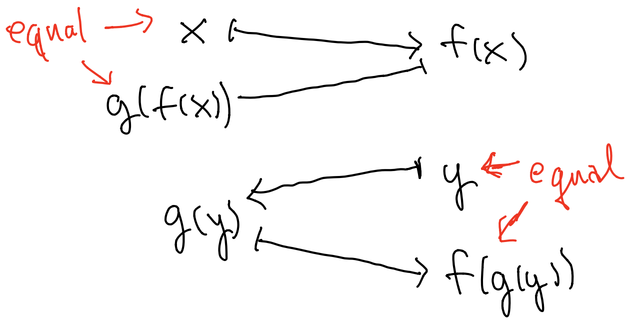
$$\Rightarrow x > 2 \text{ or } x < -1$$

$$\therefore D_{g \circ f} = (-\infty, -1) \cup (2, \infty)$$

Inverse of a function

> Suppose $f: A \to B$, $g: B \to A$
>
> Then f, g are said to be inverse of each other if
>
> $$(g \circ f)(x) = x \text{ for any } x \in A$$
>
> $$(f \circ g)(y) = y \text{ for any } y \in B$$
>
> We write $f^{-1} = g$ and $g^{-1} = f$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$



eg $f: (0, \infty) \to (0, \infty)$ $f(x) = x^2$

$g: (0, \infty) \to (0, \infty)$ $g(x) = \sqrt{x}$

For any $x, y \in (0, \infty)$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} \overset{\because x > 0}{=} x$$

$$(f \circ g)(y) = f(g(y)) = f(\sqrt{y}) = (\sqrt{y})^2 = y$$

$$\Rightarrow f^{-1} = g \text{ and } g^{-1} = f$$

Rmk If $x < 0$, then

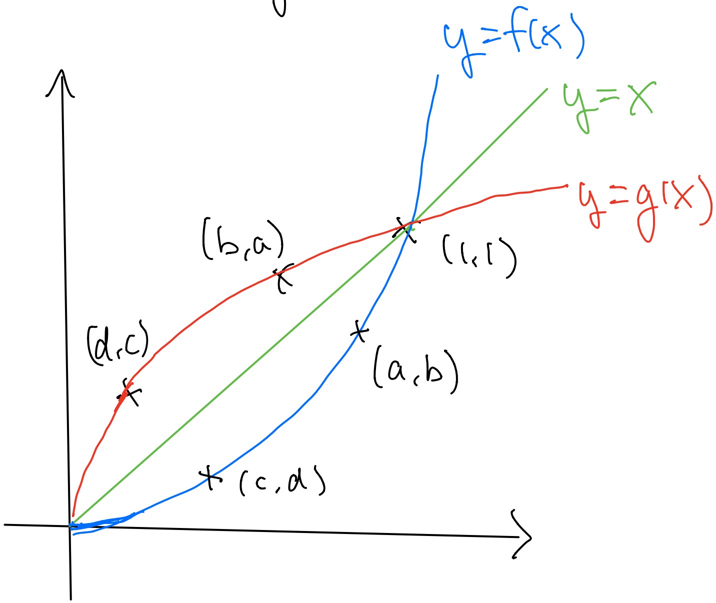
$$(g \circ f)(x) = \sqrt{x^2} = -x \neq x$$

We need to be careful about domain when we talk about inverses

$(\sqrt{x^2} = |x| \text{ for any } x \in \mathbb{R})$

Graphs of Inverses

$f(x) = x^2 \qquad g(x) = \sqrt{x}$



$\boxed{\text{Graphs of inverse functions are mirror images of each other about the line } y=x}$

Reason: (a,b) on $y=f(x)$

$\Rightarrow b = f(a)$

$\Rightarrow g(b) = a$

$\Rightarrow (b,a)$ on $y = g(x)$

Ex Think about the graphs of:

① e^x and $\ln x$

② $\sin x$ and arcs

Some Elementary functions

Piecewisely defined function

eg.
$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Find the value of

a. $f(1)$

b. $f(-2)$

c. $f(-1+h)$, where $|h| < 1$

Sol.

a. $1 \geq 0 \Rightarrow f(1) = \sqrt{1} = 1$

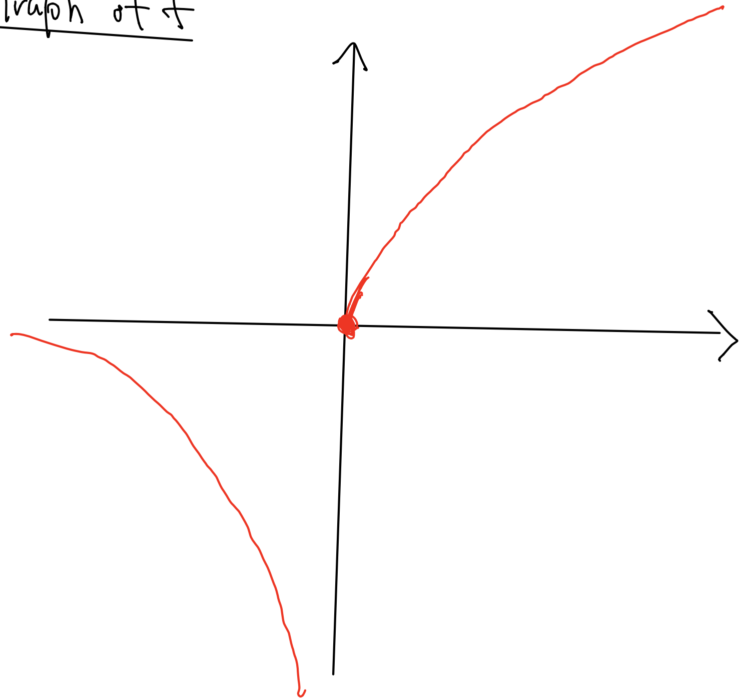
b. $-2 < 0 \Rightarrow f(-2) = \frac{1}{-2} = -\frac{1}{2}$

c. $|h| < 1 \Rightarrow -1 < h < 1$

$\Rightarrow -2 < -1+h < 0$

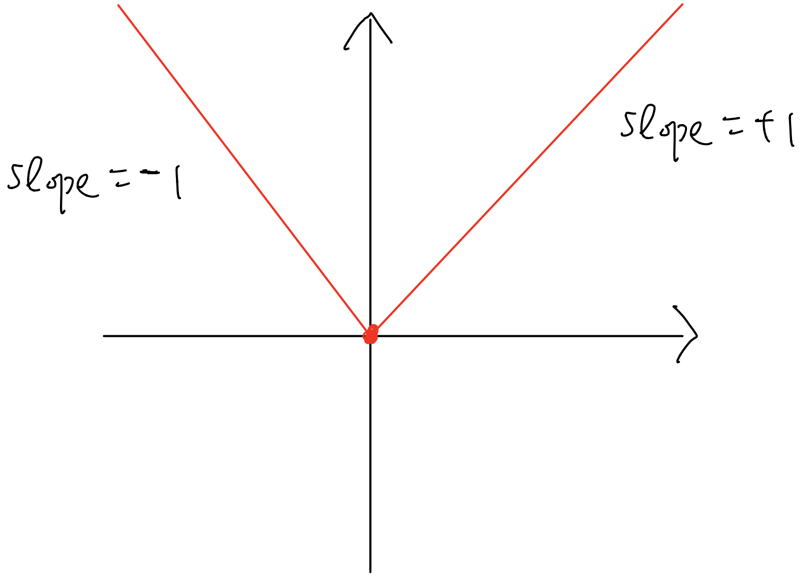
$\Rightarrow f(-1+h) = \frac{1}{-1+h}$

Graph of f



Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



e.g.

$3 \geq 0$

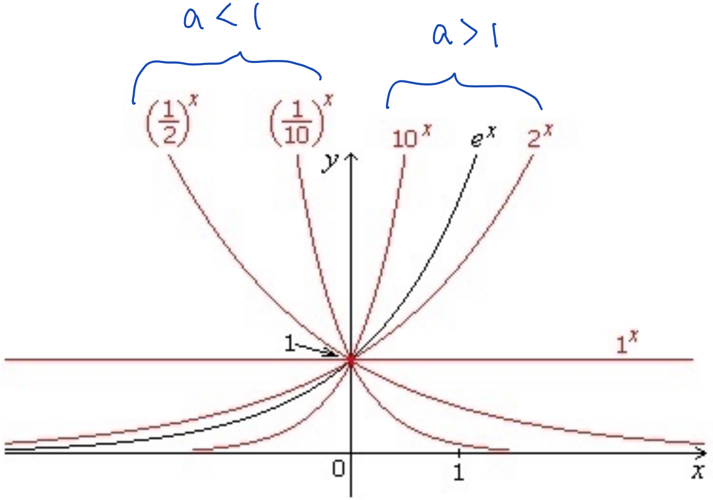
$\Rightarrow |3| = 3$

$-4 < 0$

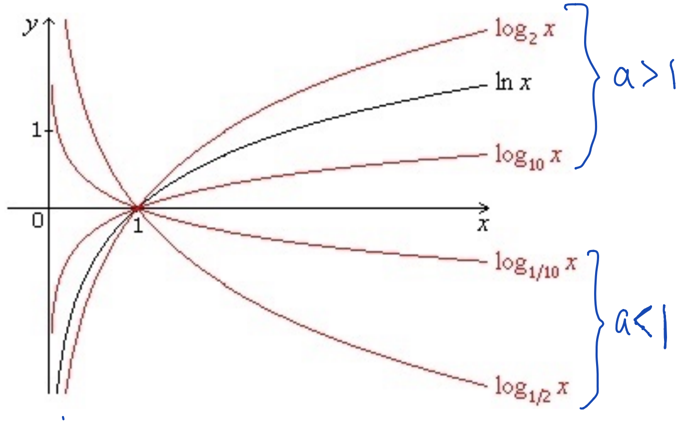
$\Rightarrow |-4| = -(-4) = 4$

Exponential and Logarithm functions

$\boxed{\text{Let } a > 0}$



$$y = a^x$$

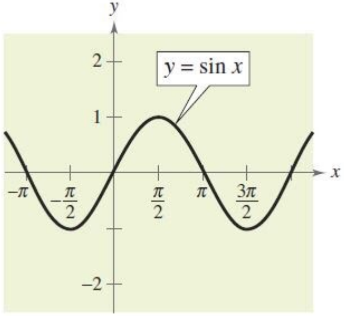


$$y = \log_a x$$

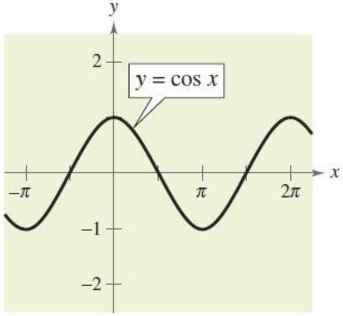
They are inverses of each other

Their graphs are mirror images of each other across the line $y = x$

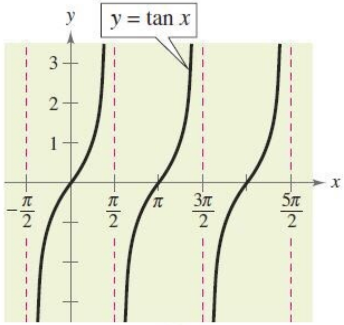
Trigonometric Functions



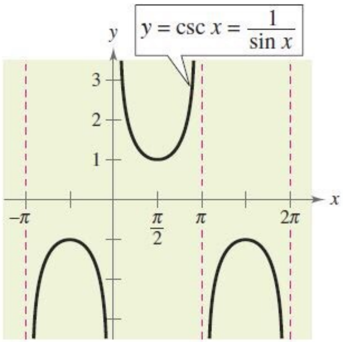
DOMAIN: $(-\infty, \infty)$
RANGE: $[-1, 1]$
PERIOD: 2π



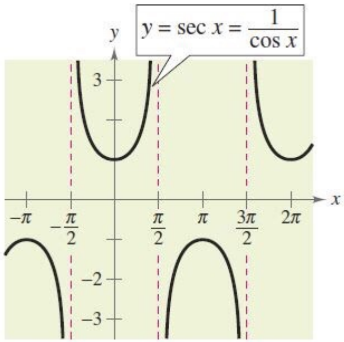
DOMAIN: $(-\infty, \infty)$
RANGE: $[-1, 1]$
PERIOD: 2π



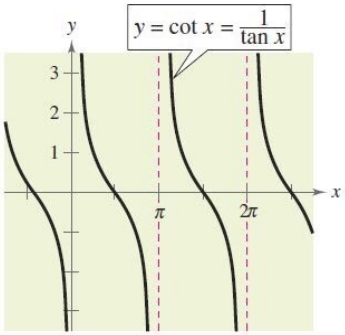
DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π

Rmk: All "angles" here are measured in radian $(180^\circ = \pi \text{ rad})$

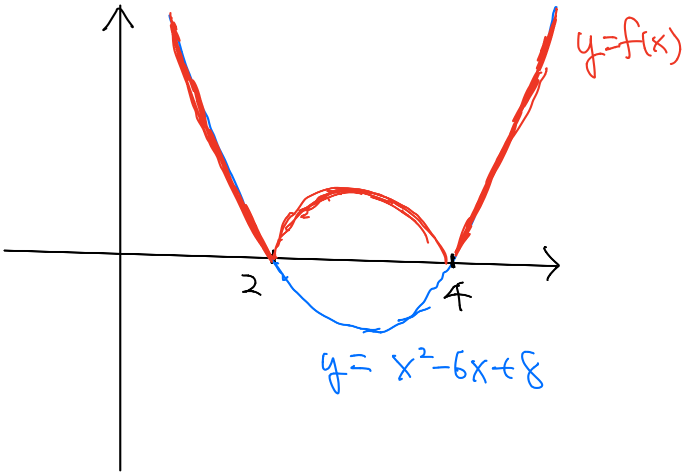
Graphing from transformation

eg Graph the following functions

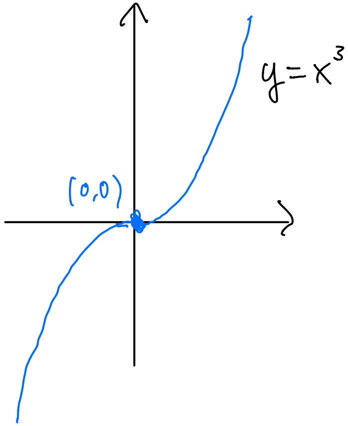
① $f(x) = |x^2 - 6x + 8|$

② $g(x) = (x-1)^3 + 2$

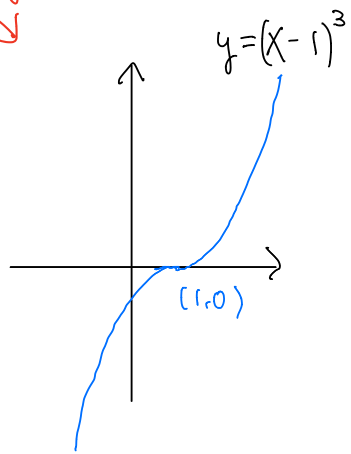
Sol ① $x^2 - 6x + 8 = (x-2)(x-4)$



② $y = x^3 \rightsquigarrow y = (x-1)^3 \to y = (x-1)^3 + 2 = g(x)$



shift to the right by 1 unit



shift up by 2 units